On the superselection sectors of fermions

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We classify elementary particles according to their behaviour under the action of the full inhomogeneous Lorentz group. For fundamental fermions, this approach leads us to delineate fermions into eight basic families or 'types', corresponding to the eight simply connected double covering groups of the inhomogeneous Lorentz group (the 'pin' groups). Given this classification, it is natural to ask whether or not fermion type determines a superselection rule. It is also important to determine what observable effects fermion type might have; for example, can the type of a given fermion be determined by laboratory experiments? We address these questions by arguing that if multiple fermion types really did occur in nature, then it would be mathematically equivalent and also much simpler to think of the different types as being different states of a *single* particle, which would be a particle which lived in the direct sum of Hilbert spaces associated with the different particle types. In the language of group theory, these are pinor supermultiplets. We discuss the possible experimental ramifications of this proposal. In particular, following work of J. Giesen, we show that the symmetries of the electric dipole moment of a particle would be definitely affected by this proposal. In fact, we show that it would be possible to use the electric dipole moment of a particle to determine the type. We also present an argument that M-theory may provide the mechanism which selects a *unique* pin bundle.

1. Introduction

The idea of a 'superselection rule' in quantum mechanics has a long and distinguished history [1]. In general, such a rule allows one to decompose some Hilbert space of states, $\mathcal{H}$, into a direct sum of subspaces $\mathcal{H}_i$ (called 'superselection sectors'): $\mathcal{H} = \bigoplus_i \mathcal{H}_i$, such that the superposition principle holds in each $\mathcal{H}_i$, but such that a linear combination, $\alpha\psi_1 + \beta\psi_2$, of states ψ_1 and ψ_2 from distinct superselection sectors is not physically realisable, except as a mixture with density matrix

$$|\alpha|^2\, \psi_1 \otimes \overline{\psi}_1 + |\beta|^2\, \psi_2 \otimes \overline{\psi}_2$$

A simple example of an observable which determines a superselection rule is given by the operator $(-1)^F$, which is even for states of integer spin (bosons) and odd for states of half-integer spin (fermions). Clearly, given a fermionic state ψ_f and a bosonic state ψ_b, we can assign no physical meaning to the linear combination $\alpha\psi_f + \beta\psi_b$. For consider the action of $R_{2\pi}$ (rotation in space through 2π about any axis) on such a state:

$$R_{2\pi}(\alpha\psi_f + \beta\psi_b) = -\alpha\psi_f + \beta\psi_b$$

Since $R_{2\pi}$ must map any physical state to an indistinguishable state, it follows that we must take $\alpha = 0$ or $\beta = 0$, i.e., it is impossible to superimpose bosons and fermions.

In this paper, we address the issue of whether or not it is possible to define superselection sectors of fermions in terms of the definitions of discrete transformations such as P and T. More precisely, there is always some ambiguity in how one defines P and T corresponding to the ambiguity in sign: $P^2 = \pm 1$, $T^2 = \pm 1$, $(PT)^2 = \pm 1$. Traditionally, it has been argued that a choice of signs for P^2, T^2, and $(PT)^2$ determines a distinct superselection sector of fermions, each sector corresponding to a different 'type' of elementary particle. Here, we discuss how one might go about forming coherent superpositions of fermionic states of different 'type'. This is achieved through a new construction of 'type-doubling', i.e., increasing the dimensions of the fermions to accomodate different type states simultaneously. Each fermion type is then just a state in a higher dimensional multiplet.

Before proceeding with this construction, however, it is useful to review some basic mathematical facts and terminology.

2. Fermions, Pin Groups, and Discrete Transformations

The study of fermions begins with the Dirac equation:

$$(i\gamma^{\mu}\,\partial_{\mu} - m)\,\psi = 0 \tag{1}$$

Dirac derived (1) by taking the square root of the standard relativistic energy-momentum relation, and making the canonical substitutions of momenta for differential operators: $p_{\mu} \rightarrow i\,\partial_{\mu}$. Dirac found that the equation could only be satisfied if the γ^{μ}s were actually 4×4 *matrices* satisfying precisely the Clifford algebra relation:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$$

where $g^{\mu\nu}$ was (for Dirac) the flat Minkowski space metric. Thus, the actual wavefunction ψ representing the electron is a *four-component* object and we are led naturally to the concept of antiparticles.

Once we form the set of solutions to equation (1) (and put an inner product structure '$<,>$' on that space so that it becomes a Hilbert space, denoted $\mathcal{H}$), it is natural to consider the representation of discrete geometrical transformations on $\mathcal{H}$. Because the nature of these representations, in their most general form, is a core issue in this paper, we feel it is probably useful to include a brief digression on the representations of a group on a vector space (in this case, a Hilbert space). To this end, let (M, g) be our underlying spacetime manifold, and let A be some group of coordinate transformations on (M, g). This group could be some global group of isometries (if the manifold admits a circle action for example) or it could be the local orthogonal group induced pointwise by the metric structure. Suppose that there exists a collection of maps, $\{O(a_i)|\forall\ a_i \in A\}$, with the property that at each $a_i \in A$, $O(a_i)$ *is a linear operator on some Hilbert space* $\mathcal{H}$. Then we say that the collection of linear maps $O(A) = \{O(a_i)|a_i \in A\}$ forms a *representation* of the group A on the Hilbert space $\mathcal{H}$ if the group structure is preserved, i.e., if $O(a_i)\,O(a_j) = O(a_i\,a_j)$, for all a_i, $a_j \in A$. Such a representation is said to be *unitary* if the corresponding maps $O(a_i)$ are unitary operators on $\mathcal{H}$. A subspace $\mathcal{H}_1 \subset \mathcal{H}$ is called *invariant* if $\forall\ v \in \mathcal{H}_1$, $O(a_i)\,v \in \mathcal{H}_1$ for any $a_i \in A$. A representation is also said to be *reducible* if there exists an invariant subspace $\mathcal{H}_1 \neq \mathcal{H}$ whose orthogonal complement $\mathcal{H}^{\perp}$ is also invariant. Otherwise, the representation is said to be *irreducible*.

Of course, a set of linear operators on a vector space is itself often a vector space. We can therefore talk about 'representing' the geometrical symmetries of A on the space $M(\mathcal{H})$ = "the set of all linear operators on $\mathcal{H}$". Clearly, $M(\mathcal{H})$ contains all of the observables in our theory. For example, let H denote a time-independent Hamiltonian. Then we say that a geometrical transformation $a \in A$ is a *symmetry* if $O(a)\,H\,(O(a))^{-1} = H$, i.e. if the two linear operators $O(a)$ and H commute.

Now, in this paper we are going to introduce operators which are not unitary; in fact, we are going to follow Wigner [17] and represent time reversal as an *anti-unitary* operator. Recall that an operator O is defined to be anti-unitary and antilinear if for any two states ϕ and ψ of the system

$$< O\phi \mid O\psi > \;\;=\;\; < \phi \mid \psi >^{*} \;\;=\;\; < \psi \mid \phi >$$

and

$$O\,(a \mid \phi > + \, b \mid \psi >) \;\;=\;\; a^{*}O \mid \phi > + \, b^{*}O \mid \psi >$$

Ordinarily, the time reversal operator is chosen to be anti-unitary in order to insure that positive energy states are mapped to positive energy states. Since the product of a unitary operator and an anti-unitary operator is anti-unitary, and parity inversion is unitary, it follows that the combined operation of parity inversion with time reversal is anti-unitary. This state of affairs will hold for all of the operators which we write down in this paper, i.e., time reversal and the combined operation of parity inversion with time reversal will always be

anti-unitary. This choice is the standard choice made in the particle physics literature; DeWitt-Morette et al [8] refer to this choice as the physical or 'non-relativistic' choice. In many books, a representation is *defined* to be a representation of a group by *unitary* operators. Thus, in this sense we are not truly considering irreducible *representations* of the inhomogeneous Lorentz group in this paper. On the other hand, we are considering what Wigner ([17], page 335) refers to as '*corepresentations*'; a corepresentation is just like a unitary representation only some of the operators are allowed to be anti-unitary. Clearly, a corepresentation is mathematically distinct from a representation, and so it is very important not to confuse the two things (this point is emphasized in [8]). Technically, then, this paper is concerned strictly with corepresentations of the inhomogeneous Lorentz group.[1]

The above discussion is very general and can be applied in a wide range of situations. We now wish to specialise and concentrate our attention on the one group which will survive in any field theory which incorporates relativistic covariance with discrete transformations: the inhomogeneous Lorentz group, $O(3,1)$.

The best way to illustrate what we are talking about is with an explicit example. Let us therefore recall how the operators C (charge conjugation), P (parity inversion) and T (time reversal) are represented in the particle physics literature [7]: Let $\mathcal{H}$ be the set of solutions of the Dirac equation on four-dimensional Minkowski space; then C, P, and T are operators on $\mathcal{H}$ given by the explicit formulae:

$$\begin{aligned} C &: \psi(x,t) \to i\gamma^2\psi^*(x,t) \\ P &: \psi(x,t) \to \gamma^0\psi(-x,t) \\ T &: \psi(x,t) \to \gamma^1\gamma^3\psi^*(x,-t) \end{aligned} \tag{2}$$

where ψ is any solution and * denotes the operation of complex conjugation. We remind the reader (without going into details) that a host of physical considerations goes into the choices made in equations (2). A number of other choices are possible, the key point being that the other choices are *mathematically inequivalent.*

Now, one of the first things we can notice about the operators P and T defined in (2) is that they do not give a *Cliffordian* representation of the action of space and time inversion. That is, P and T do not anti-commute, since in fact they commute:

$$PT \sim \gamma^0\gamma^1\gamma^3 = \gamma^1\gamma^3\gamma^0 \sim TP$$

Therefore, the operators P and T defined in (2) correspond to a *non-Cliffordian* representation of $O(3,1)$ with non-Cliffordian *action.*

This situation can be contrasted with the case where the representation has Cliffordian action. For example, a Cliffordian action can be recovered by the following operator assignment:

$$\begin{aligned} P &: \psi(x,t) \to \gamma^1\psi(-x,t) \\[1ex] T &: \psi(x,t) \to \gamma^0\psi(x,-t) \end{aligned} \tag{3}$$

Clearly, the (unitary) choices in (3) anti-commute.

Of course, in each of the above examples, the underlying group structure is identical. More precisely, in the operator assignments made in (2), we used the group of elements γ^μ satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ to construct operators P and T whose *action* on $\mathcal{H}$ is non-Cliffordian, whereas in (3) we used the *same group* of Cliffordian elements to construct operators P and T with *Cliffordian* action. It is absolutely essential that we make this distinction between the different actions on a Hilbert space which can be constructed from a given group, and genuinely *different groups.* This is because we are sympathetic to the philosophy of Wigner [4] who put forward the idea that the irreducible (co)representations of whatever group of symmetries is present in nature should form the basis for any theory of elementary particles. Indeed, Wigner completely classified the set of irreducible

[1] See also [20] for another discussion of these issues.

corepresentations of the inhomogeneous Lorentz group, $O(3,1)$, on the Hilbert space of solutions to the Dirac equation (1) with $m \neq 0$. He showed that once one 'fixes' the sign of the square of parity inversion P^2 (fixing this sign corresponds to choosing a signature for spacetime, basically) then there are four *inequivalent* (non-isomorphic) cases. The first case is the standard particle physics choice made in (2) above. In the remaining three cases, there is a phenomenon known as 'parity doubling', which can be described as follows.

To begin with, there are simply not enough choices possible, when the dimension of the corepresentation is 4, to realise all of the irreducible corepresentations. That is to say, if we stick with only using 4×4 matrices to write P and T as linear operators on $\mathcal{H}$, then we can really only use combinations of the γ^μs and so we are stuck with the standard Cliffordian *group* which we used in examples (2) and (3) above. We therefore need to somehow increase the dimension of our corepresentation, and in fact this is exactly what Wigner did when he showed how to obtain the remaining three corepresentations by *doubling* the dimension.

Explicitly, what one first does is write down the 'doubled' gamma matrices, Γ^μ (the 'big' gammas) as follows:

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix} \tag{4}$$

The 'doubled' Dirac equation then becomes

$$(i\Gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{5}$$

Thus, solutions to (5) are now *eight* component 'pinor' fields. Intuitively, one can now think of the extra degrees of freedom in the solutions of (5) as corresponding to the assignment of 'parity'.

We can now obtain the non-standard irreducible corepresentations of $O(3,1)$ by representing P and T on the set of solutions, $\mathcal{H}_D$ (the 'doubled' Hilbert space), to (5). Of course, this might seem confusing since although the Γ^μs are eight component matrices, they still satisfy

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$$

The point is, we are *no longer bound* to only use combinations of the Γ^μs to construct our corepresentations. The only thing [4] which distinguishes the different irreducible corepresentations (once we have fixed the signature) is the *sign of the squares of the operators representing T and PT*. Let us fix the signature to be (for now) $(-+++)$. Then the sign of parity inversion squared is fixed (in all the corepresentations) to be

$$P^2 = -\mathrm{Id}$$

Thus, in the 'standard' case presented above (which we shall denote *Case I*) $P^2 = \gamma^0\gamma^0 = -\mathbb{I}$, $T^2 = \gamma^1\gamma^3\gamma^1\gamma^3 = -\mathbb{I}$, $(PT)^2 = \gamma^0\gamma^1\gamma^3\gamma^0\gamma^1\gamma^3 = \mathbb{I}$ where $\mathbb{I} = \mathrm{Id}$ is the identity matrix. The other three cases can therefore be presented as follows.

Case II: Here, we seek operators P, T, and PT on the space of solutions $\mathcal{H}_{\mathcal{D}}$ to (5) such that $P^2 = -\mathbb{I}$, $T^2 = -\mathbb{I}$, and $(PT)^2 = -\mathbb{I}$. Such a corepresentation is given by the following assignments:

$$\begin{aligned} P : \psi(x,t) &\to \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix} \psi(-x,t) \\ T : \psi(x,t) &\to \begin{pmatrix} 0 & \gamma^1\gamma^3 \\ \gamma^1\gamma^3 & 0 \end{pmatrix} \psi^*(x,-t) \end{aligned} \tag{6}$$

Case III: Here, we seek operators P and T such that $P^2 = -\mathbb{I}$, $T^2 = +\mathbb{I}$, and $(PT)^2 = +\mathbb{I}$. Such a corepresentation is given by the following assignments:

$$
\begin{aligned}
P &: \psi(x,t) \rightarrow \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix} \psi(-x,t) \\
T &: \psi(x,t) \rightarrow \begin{pmatrix} 0 & \gamma^1\gamma^3 \\ -\gamma^1\gamma^3 & 0 \end{pmatrix} \psi^*(x,-t)
\end{aligned} \tag{7}
$$

Case IV: Finally, in this case we seek operators P and T for which $P^2 = -\mathbb{I}$, $T^2 = +\mathbb{I}$, and $(PT)^2 = -\mathbb{I}$. This is accomplished by the following definitions:

$$
\begin{aligned}
P &: \psi(x,t) \rightarrow \begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 \end{pmatrix} \psi(-x,t) \\
T &: \psi(x,t) \rightarrow \begin{pmatrix} 0 & \gamma^1\gamma^3 \\ -\gamma^1\gamma^3 & 0 \end{pmatrix} \psi^*(x,-t)
\end{aligned} \tag{8}
$$

Of course, if we change the signature (or just the sign of P^2) then we again obtain four inequivalent corepresentations. These eight different ways of writing the operations P and T thus correspond to eight different non-isomorphic groups. These groups are called the *pin groups*, and it is time we turned our attention to formally defining them.

To this end, recall that generally we do physics on *spacetimes*, M, which may not necessarily be orientable. What this means is that the tangent bundle, τ_M, can at most be reduced to an $O(p,q)$ bundle. When the metric, g_{ab}, has signature $(-+++)$ then the structure group will be $O(3,1)$. When the metric has signature $(+---)$ then the structure group will be $O(1,3)$ (actually, $O(3,1) \simeq O(1,3)$, but as we shall see it is necessary to keep the distinction when we pass to the double covers). Since $\pi_1(O_0(3,1) \simeq \pi_1(O_0(1,3)) \simeq \mathbb{Z}_2$, we are interested in finding all groups which are double covers of $O(3,1)$ and $O(1,3)$. There are *eight* distinct such double covers [6] of $O(p,q)$. Following Dąbrowski, we will write these covers as

$$
h^{a,b,c} : \ \mathrm{Pin}^{a,b,c}(p,q) \longrightarrow O(p,q)
$$

with $a,b,c \in \{+,-\}$. The signs of a,b, and c can be interpreted in the following way:

Recall, first, that $O(p,q)$ is not path connected; there are four components, given by the identity connected component, $O_0(p,q)$, and the three components corresponding to parity reversal P, time reversal T, and the combination of these two, PT (i.e., $O(p,q)$ decomposes into a semidirect product[2], $O(p,q) \simeq O_0(p,q) \odot (\mathbb{Z}_2 \times \mathbb{Z}_2)$). The signs of a,b, and c then correspond to the signs of the squares of the elements in $\mathrm{Pin}^{a,b,c}(p,q)$ which cover space reflection, R_S, time reversal, R_T and a combination of the two respectively. That is, in this paper we adopt precisely the following convention:

$$
\begin{aligned}
P^2 &= a \\
T^2 &= b \\
(PT)^2 &= c
\end{aligned}
$$

We note that this convention differs markedly from Dąbrowski, who takes

$$
\begin{aligned}
P^2 &= -a \\
T^2 &= b \\
(PT)^2 &= -c
\end{aligned}
$$

[2]That is, $O(p,q)$ is the disjoint union $O(p,q) = (O_0(p,q)) \ \cup \ P(O_0(p,q)) \ \cup \ T(O_0(p,q)) \ \cup \ PT(O_0(p,q))$, and the four element group $\{1,P,T,PT\}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(In our notation, the obstruction theory is more transparent, although there are other reasons for adopting Dąbrowski's notation).

With this in mind we can, following Dąbrowski [6], write out the explicit form of the groups $\mathrm{Pin}^{a,b,c}(p,q)$; they are given by the semidirect product

$$\mathrm{Pin}^{a,b,c}(p,q) \simeq \frac{(\mathrm{Spin}_0(p,q) \odot C^{a,b,c})}{\mathbb{Z}_2}$$

where the $C^{a,b,c}$ are the four double coverings of $\mathbb{Z}_2 \times \mathbb{Z}_2$; i.e., $C^{a,b,c}$ are the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (when $a = b = c = +$), D_4 (dihedral group, when there are two plusses and one minus in the triple a, b, c), $\mathbb{Z}_2 \times \mathbb{Z}_4$ (when there are two minuses and one plus in a, b, c), and Q_4 (quaternions, when $a = b = c = -$).

Clearly, the different pin groups correspond to the different ways of defining the operators P and T. We shall therefore say that the different pin groups determine different *types* of fermions. Our goal now is to explore the extent to which fermion type defines a superselection rule, i.e., is it possible to form a coherent superposition of fermions of different type?

3. Fermion Type and Superselection

Traditionally, people have assumed that fermion type determines a superselection rule, i.e., that it is impossible to form a linear combination of fermionic states of differing type. This prejudice is based primarily on the fact that the different pin groups form all of the *irreducible* representations of the inhomogeneous Lorentz group. Thus, any attempt to mix fermions of differing type will require passing to a manifestly non-irreducible representation.

In order to rigorously see why fermion type determines a superselection rule, it would be nice if we could write down an equation similar to the one used in the introduction to show that the observable $(-1)^F$ yields superselection. To do this, let $\mathcal{H}^{a,b,c}$ denote the Hilbert space for a particle of type (a, b, c) acted on by $\mathrm{Pin}^{a,b,c}(3, 1)$. Let $P_{(a,b,c)}$ and $T_{(a,b,,c)}$ denote the operations of parity and time reversal in $\mathrm{Pin}^{a,b,c}(3, 1)$. Consider two fermions of distinct type, $\psi_+ \in \mathcal{H}^{+,b,c}$ and $\psi_- \in \mathcal{H}^{-,b,c}$. We want to know if it makes sense to form the linear combination $\alpha\psi_+ + \beta\psi_-$. Naïvely then, we want to consider an expression of the form

$$P^2(\alpha\psi_+ + \beta\psi_-)$$

However, an obvious problem which presents itself is: Which 'P' do we choose? Clearly, it makes no sense mathematically to have either $P = P_{(+,b,c)}$ *or* $P = P_{(-,b,c)}$. It *does* make sense to write

$$P = \begin{pmatrix} P_{(+,b,c)} & 0 \\ 0 & P_{(-,b,c)} \end{pmatrix}$$

and to think of ψ_+ and ψ_- as two 'states' of a 'larger' particle Φ:

$$\Phi = \begin{pmatrix} \alpha\psi_+ \\ \beta\psi_- \end{pmatrix}$$

In fact, not only does this construction make sense, it is mathematically justified; to see this, consider the following thought experiment:

Suppose we are given a system consisting of two particles of type (a, b, c) and two particles of different type (a', b', c'). Then the appropriate Hilbert space for such a system is

$$\left(\mathcal{H}^{a,b,c} \circledA \mathcal{H}^{a,b,c}\right) \oplus \left(\mathcal{H}^{a,b,c} \otimes \mathcal{H}^{a',b',c'}\right) \oplus \left(\mathcal{H}^{a',b',c'} \circledA \mathcal{H}^{a',b',c'}\right) \tag{9}$$

where Ⓐ denotes antisymmetric product, $\oplus$ denotes direct sum and $\otimes$ denotes tensor product. (In other words, the pair of (a, b, c) particles and the pair of (a', b', c') particles each satisfy Pauli exclusion since they are each

pairs of identical particles). The beautiful thing is that the Hilbert space in Equation (9) is actually *isomorphic* to the Hilbert space

$$\left(\mathcal{H}^{a,b,c} \oplus \mathcal{H}^{a',b',c'}\right) \textcircled{A} \left(\mathcal{H}^{a,b,c} \oplus \mathcal{H}^{a',b',c'}\right)$$

In other words, it is mathematically equivalent to think of the four particle system as a *two* particle system consisting of two fermions, each living in the Hilbert space $\left(\mathcal{H}^{a,b,c} \oplus \mathcal{H}^{a',b',c'}\right)$. We shall refer to fermions which live in such direct sum Hilbert spaces as 'mixed' fermions or *meta*-fermions. In the language of group theory these objects are *pinor supermultiplets*, since each 'state' of the multiplet is an object corresponding to a distinct pin group; we emphasize that this use of the word 'supermultiplet' has nothing to do with supersymmetry, i.e., we are using the terminology of Chapter 18 of [19]. Thus, by passing to the space of mixed fermions we can considerably simplify the mathematical structure of a problem (although we are still dealing with the same amount of information). Of course, in general there will be eight (not just two) types of fermion present; suppose that the total number of fermions (of whatever type) is N. Then the generalisation of the above Hilbert space isomorphism implies that we can always think of such a system as consisting of N identical particles, each living in the Hilbert space

$$\bigoplus_{(a,b,c)\in\{\pm\}} \mathcal{H}^{a,b,c}$$

In other words, the general Hilbert space for fermions is

$$\underbrace{\left(\bigoplus_{(a,b,c)\in\{\pm\}} \mathcal{H}^{a,b,c}\right) \textcircled{A} \left(\bigoplus_{(a,b,c)\in\{\pm\}} \mathcal{H}^{a,b,c}\right) \textcircled{A} \dots \textcircled{A} \left(\bigoplus_{(a,b,c)\in\{\pm\}} \mathcal{H}^{a,b,c}\right)}_{N \text{ times}} \tag{10}$$

Clearly, this proposal is very similar to Heisenberg's old suggestion [9] that we should think of the proton p and the neutron n as two 'states' of a single particle, the nucleon N:

$$N = \begin{pmatrix} p \\ n \end{pmatrix}$$

Of course, Heisenberg took things further, introducing the abstract 'isospin space', defining the proton to be isospin up and the neutron to be isospin down, and proposing that strong interaction physics is invariant under rotations in isospin space. In other words, in terms of group theory, he asserted that strong interactions are invariant under the action of an internal symmetry $SU(2)$, and that nucleons determine a two-dimensional representation (i.e., they are isospin $\frac{1}{2}$). This proposal, which was motivated by the simple fact that strong interactions do not distinguish between protons and neutrons, had far-reaching consequences.

To our knowledge, *none* of the four forces distinguish between fermions because of type; indeed, the only 'physical' effect of fermion type known to us (we will discuss this in more detail later) is the fact [5] that some types of fermions *do not* have CP-violating electric dipole moments whereas other types do. Given this, it is tempting to *conjecture* that any physics involving the mixed fermion supermultiplet which we constructed above is invariant under the maximal internal symmetry group $U(8)$. If this were true, then the supermultiplet would form a *fundamental* (eight-dimensional) representation of $U(8)$. On the other hand, it may be that some physical processes break the symmetry down to some $(S)U(n)$, $n < 8$. We simply cannot tell since we have no real experimental data which determines fermion type and, more seriously, we do not even know if there is more than one type of fermion in the universe. Nevertheless, it is amusing to take these abstract group-theoretic conjectures seriously and see if they might lead us to any real physics; this avenue of research is currently being actively investigated. We will have more to say about the possible experimental consequences of this proposal that fermions live in eight-dimensional supermultiplets later.

We conclude this section with a sketch of the structure which we have proposed:

A fermion Ψ generically lives in a direct sum of Hilbert spaces $\mathcal{H}_{a,b,c}$, where each $\mathcal{H}^{a,b,c}$ is acted on by a representation of the relevant pin group $\mathrm{Pin}^{a,b,c}(p,q)$. Explicitly, Ψ looks like this:

$$\Psi = \begin{pmatrix} \Psi_{+++} \\ \Psi_{+-+} \\ \Psi_{+--} \\ \vdots \\ \Psi_{---} \end{pmatrix} \tag{11}$$

We emphasize that this is only a *proposal*. It may well be the case that every electron (for example) in the universe lives in a Hilbert space acted on by just one of the pin groups. If this turns out to be the case, then the hypothesis of pinor multiplets is a needless complication. On the other hand, it may well be the case that some electrons are of type $(+,+,+)$, whereas other electrons are of type $(+,-,+)$, and so forth. If this turns out to be the case, then we have to assign extra internal quantum numbers (namely the three signs for a, b and c) to any electron in order to completely classify the state.

Ψ is acted upon by a 'total' parity, or metaparity operator, P, which also is a direct sum of the individual parity operators $P_{(a,b,c)}$ coming from each $\mathrm{Pin}^{a,b,c}(p,q)$, i.e.,

$$P = \begin{pmatrix} P_{(+++)} & & & & \\ & P_{(+-+)} & & \text{\Large 0} & \\ & & P_{(+--)} & & \\ & \text{\Large 0} & & \ddots & \\ & & & & P_{(---)} \end{pmatrix} \tag{12}$$

Similarly, there is a total time inversion, T, which looks like

$$T = \begin{pmatrix} T_{(+++)} & & & & \\ & T_{(+-+)} & & \text{\Large 0} & \\ & & T_{(+--)} & & \\ & \text{\Large 0} & & \ddots & \\ & & & & T_{(---)} \end{pmatrix} \tag{13}$$

and similarly for PT.

In the absence of any interaction, propagation in each Hilbert space is given by the ordinary Dirac equation. This would seem to justify the supposition that the different $\mathcal{H}^{a,b,c}$ determine superselection sectors for fermions, i.e., that different fermion types cannot interfere. However, it is likely that an argument similar to the one given by Aharonov and Susskind [3] can be constructed to explicitly show how to prepare states which are coherent superpositions of fermions of differing types. Recall that Aharonov and Susskind presented a thought experiment, which could be *performed* in principle, in which they showed how to prepare a state which is a coherent superposition of a proton and a neutron:

$$\alpha p \, + \, \beta n$$

Since the proton and neutron components of this nucleon can interfere, this amounts to a violation of the charge superselection rule. It is likely that a similar thought experiment can be conceived for fermion type; it is probably just a question of understanding how to distinguish (in the lab) and isolate, fermions of differing types.

With this in mind, we now turn to a discussion of what observable consequences (if any) follow from our proposal that real fermions actually belong to these eight-dimensional pinor supermultiplets.

4. CP-Invariance and Electric Dipole Moments

A great deal of experimental evidence has been amassed which establishes very strong bounds on the electric dipole moments of various elementary particles [11], [12]. In particular, it has been shown that the electric dipole moment (e.d.m.) of the electron (denoted d_e) satisfies [11]

$$d_e < (-0.3 \pm 0.8) \times 10^{-26} \text{ ecm} \tag{14}$$

and that the e.d.m. of the neutron (denoted d_n) satisfies

$$d_n < 11 \times 10^{-26} \text{ ecm} . \tag{15}$$

Clearly, these bounds imply that the e.d.m.'s of these particles are *extremely* small, even smaller than the particles themselves ($\sim 10^{-13}$ cm for the neutron n). It is very important that we know the *precise* value of d_n since it is related to other quantities which arise naturally in the standard model (SM).

For example, the $SU(3) \times SU(2) \times U(1)$ gauge sector of SM has a non-trivial vacuum structure [13]. This vacuum structure gives rise to phases (or 'θ-vacua' [14]) which imply the existence of CP-violating effective interaction terms, which involve the non-Abelian gauge fields:

$$\mathcal{L}_{\text{eff}} \simeq \theta_s \frac{\alpha_s}{8\pi} F_a^{\mu\nu} \tilde{F}_{a\mu\nu} + \theta_w \frac{\alpha_w}{8\pi} W_a^{\mu\nu} W_{a\mu\nu}$$

Since the electroweak theory is chiral, we can always rotate the *weak* vacuum angle, θ_w, to zero. However, the *strong* vacuum angle θ_s is more complicated; one has to perform chiral rotations that leave the quark mass matrices diagonal. This means that θ_s receives corrections from the weak sector:

$$\bar{\theta} = \theta_s + \arg(\det M)$$

where M is the quark mass matrix. In other words, the *physical* CP-violating interaction is

$$\mathcal{L}_{\text{CP}} = \bar{\theta} \frac{\alpha_s}{8\pi} F_a^{\mu\nu} \tilde{F}_{a\mu\nu}$$

Interestingly [16], the existence of such an interaction in SM contributes substantially to the neutron e.d.m., d_n. In fact,

$$d_n \approx 8.2 \times 10^{-16} \bar{\theta} \text{ ecm} \tag{16}$$

Actually, this estimate is based on a calculation in QCD, and it *assumes* that the e.d.m. of the neutron is CP-violating and of course that the neutron is a system made up of three quarks. Perhaps the truly interesting thing to do here is to try and repeat the calculation of [16] while allowing for the quarks themselves to be particles of differing type. In this paper we are being more simple minded about things and regarding the neutron itself as an elementary particle. At any rate, given the above bound equation (17) on d_n, we see that $\bar{\theta}$ must satisfy [11]

$$\bar{\theta} \leq 10^{-9} - 10^{-10} \tag{17}$$

Finding an explanation for this phenomenon is known as the *strong CP problem.*

While we do not solve the strong CP problem here, we *do* present proof that fermions of differing type possess e.d.m.'s which break *differing* combinations of C, P, or T. More precisely, we show that by choosing different types we can construct fermions with e.d.m.'s which are *not* CP-violating, but which may be (for example) C-violating as well as P-violating (hence T-non-violating by the CPT theorem). In order to understand this construction, we need to recall the recent work of Giesen [5].

In [5] Giesen studied the behaviour of the e.d.m.'s of particles of differing types under the action of discrete space-time symmetries. What he found is that while the 'standard' four-component fermions (in the chiral

representation) of type $(+,-,-)$ possess e.d.m.'s which are both P and T violating (and hence CP violating), the 'non-standard' eight-component fermions of type $(+,+,-)$ (the $a = P^2 = +1$ analogue of Case III, equation (7) above) possess e.d.m.'s which are *neither* P nor T violating (and hence do *not* violate CP).

In order to make everything explicit, we write out the actions of C, P, and T for fermion types with $P^2 = +1$ in the below table (this table is the $a = P^2 = +1$ analogue of equations (2), (6), (7), (8) above). Throughout this section we are working in the 'chiral' representation, i.e.,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$$

and

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where the σ_i are ordinary Pauli matrices. The actions of the different discrete transformations are then given as shown:

Fermion Type	Actions of C, P, and T
$(+,-,-)$	$\left.\begin{array}{l} C : \psi(x,t) \longrightarrow i\gamma^2\psi^*(x,t) \\ P : \psi(x,t) \longrightarrow \gamma^0\psi(-x,t) \\ T : \psi(x,t) \longrightarrow \gamma^1\gamma^3\psi^*(x,-t) \end{array}\right\}$ Four-component corepresentation
$(+,-,+)$	$\left.\begin{array}{l} C : \psi(x,t) \longrightarrow i\begin{pmatrix} \gamma^2 & 0 \\ 0 & -\gamma^2 \end{pmatrix}\psi^*(x,t) \\ P : \psi(x,t) \longrightarrow \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix}\psi(-x,t) \\ T : \psi(x,t) \longrightarrow \begin{pmatrix} 0 & \gamma^1\gamma^3 \\ \gamma^1\gamma^3 & 0 \end{pmatrix}\psi^*(x,-t) \end{array}\right\}$ Doubled corepresentation (eight-component)
$(+,+,-)$	$\left.\begin{array}{l} C : \psi(x,t) \longrightarrow i\begin{pmatrix} \gamma^2 & 0 \\ 0 & \gamma^2 \end{pmatrix}\psi^*(x,t) \\ P : \psi(x,t) \longrightarrow \begin{pmatrix} \gamma^0 & 0 \\ 0 & -\gamma^0 \end{pmatrix}\psi(-x,t) \\ T : \psi(x,t) \longrightarrow \begin{pmatrix} 0 & \gamma^1\gamma^3 \\ -\gamma^1\gamma^3 & 0 \end{pmatrix}\psi^*(x,-t) \end{array}\right\}$ Doubled corepresentation (eight-component)
$(+,+,+)$	$\left.\begin{array}{l} C : \psi(x,t) \longrightarrow i\begin{pmatrix} \gamma^2 & 0 \\ 0 & -\gamma^2 \end{pmatrix}\psi^*(x,t) \\ P : \psi(x,t) \longrightarrow \begin{pmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 \end{pmatrix}\psi(-x,t) \\ T : \psi(x,t) \longrightarrow \begin{pmatrix} 0 & \gamma^1\gamma^3 \\ -\gamma^1\gamma^3 & 0 \end{pmatrix}\psi^*(x,-t) \end{array}\right\}$ Doubled corepresentation (eight-component)

Table 1

In the above table, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, with $g^{\mu\nu}$ of signature $(+,-,-,-)$ (so that $P^2 = +1$ everywhere).

To see how the e.d.m.'s of particles of different type transform, we follow [5] and write the Dirac equation for a four-component fermion ψ with dipole moment strength d coupled to an external electromagnetic field A_μ (with field strength $F_{\mu\nu}$) as follows:

$$(\gamma^\mu(i\partial_\mu + eA_\mu) - d\gamma^\mu\gamma^\nu\gamma_5 F_{\mu\nu} - m)\,\psi = 0 \tag{18}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The extra term in this otherwise minimally coupled Dirac equation comes from the addition of a gauge invariant, covariant effective Lagrangian term

$$\mathcal{L}_{\text{eff}} = -d\psi^\dagger\gamma^\mu\gamma^\nu\gamma_5 F_{\mu\nu}\psi$$

The non-relativistic limit of this coupling is the usual $\hat{\sigma}\cdot E$ type interaction, where

$$\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$$

$$\hat{\sigma}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

and σ_i are the Pauli matrices.

Let $\psi_P = P\psi$ denote the parity inversion of ψ, and $\psi_T = T\psi$ the time inversion of ψ. Then it is a standard result [5] that ψ_P is *not* a solution of the parity reflection of equation (18), and similarly ψ_T is *not* a solution of the time reflection of equation (18). Thus, reflected solutions do not solve the reflected equation; we therefore say that solutions of (18) violate P and T symmetry. This is an old result, which holds for the e.d.m.'s of all four-component fermions of type $(+,-,-)$.

However, things change considerably when we write down the equation describing a dipole moment for a *non-standard* eight-component fermion [5]:

$$\left(\Gamma^\mu(i\partial_\mu + eA_\mu) - d\begin{pmatrix} 0 & \gamma^\mu\gamma^\nu\gamma_5 \\ \gamma^\mu\gamma^\nu\gamma_5 & 0 \end{pmatrix} F_{\mu\nu} - m\right)\psi = 0 \tag{19}$$

where $\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\nu \end{pmatrix}$ are the doubled gamma matrices. Equation (19) arises by adding the effective Lagrangian term

$$\mathcal{L}_{\text{eff}} = -d\psi^\dagger\begin{pmatrix} 0 & \gamma^\mu\gamma^\nu\gamma_5 \\ \gamma^\mu\gamma^\nu\gamma_5 & 0 \end{pmatrix} F_{\mu\nu}\psi$$

In the non-relativistic limit, this coupling takes the form

$$\begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix}\cdot E$$

with $\hat{\sigma}$ as above.

For fermions of type $(+,+,-)$ Giesen showed [5] that the reflected solutions ψ_P and ψ_T *are* solutions of the P and T inversions of equation (19). Thus, all fermions of type $(+,+,-)$ possess e.d.m.'s which are *not* CP-violating.

A natural question, then, is to determine how the e.d.m's of *other* types of particles transform under the action of C, P. and T. It is not too hard to work out; the results are displayed in the below table.

Fermion Type	e.d.m. violates C?	e.d.m. violates P?	e.d.m. violates T?	e.d.m. violates CP?
$(+,-,-)$	NO	YES	YES	YES
$(+,-,+)$	YES	NO	YES	YES
$(+,+,-)$	NO	NO	NO	NO
$(+,+,+)$	YES	YES	NO	NO

Table 2

Clearly, what Table 2 provides us with is a way of in principle determining the type of an elementary particle. For suppose that you are given any elementary particle 'x' with non-vanishing e.d.m. 'd'. Then you can determine the type of x (up to the sign of P^2) simply by determining which combination of C, P and T d violates (there will be will be a table identical to Table 2 for the quartet of particles with $P^2 = -1$). To our knowledge, this is the first 'in principle' performable test for determining the type of a fermion (but see [8] for further discussion of these points). The only other example where different fermion types yield different observables was presented in [10], where it was shown that the vacuum expectation value of the fermionic current on a Klein bottle will depend crucially upon which pin structure you use to construct the fermions. Actually, we have no problem with this example since as far as we are concerned if one accepts the path integral prescription for quantum gravity then a sum over histories means a sum over *all* topologies, including non-orientable manifolds. Unfortunately, many people still have an aversion to the concept of non-orientable spacetime foam. The Giesen construction is therefore better for determining fermion type since it involves nothing more than quantum mechanics on flat spacetime.

5. Conclusion: Does M-theory select a unique pin structure?

We have attempted to determine the logical consequences of the proposal that elementary particles should be classified according to how they behave under the action of the full inhomogeneous Lorentz group. We have argued that if more than one 'type' of particle actually occurs in nature, then it is simplest to arrange the different types into 'mixed' particles, or multiplets. We have also examined and extended Giesen's work on the nature of the electric dipole moments of elementary particles of differing types. We have shown that the type of any fermion x with non-vanishing e.d.m. can be determined once one knows which combination of C, P, or T invariance x violates when it interacts with an external electromagnetic field. We have argued that the next logical thing to do is to repeat the calculation of [16] for the neutron e.d.m., allowing the quarks to be of any type.

Of course, it is not hard to see that most of the observed elementary particles can only come in one type. For example, suppose that there existed two types of electron, a 'plus' type and a 'minus' type. The Pauli exclusion principle would allow you to place a plus electron and a minus electron in the same state. Obviously, this would seriously mess up most of known chemistry unless the electromagnetic interaction coupled only to one type, and the other type was decoupled from known matter! Thus, it would seem that nature has selected a particular pin structure for the description of elementary particles. From a four-dimensional point of view, it is unclear why or how nature makes such a selection.

In the search for some mechanism which could give rise to such a selection rule, it is natural to appeal to some fundamental theory which might be valid at arbitrarily high energies. At the present time, our best hope for such a 'theory of everything' is the body of knowledge commonly referred to as 'M-theory'. While we still aren't really sure about what M-theory actually is, or what it describes in general, we are sure that the low energy limit is $D = 11$, $N = 1$ supergravity theory.

Now, $D = 11$, $N = 1$ supergravity is a theory which describes the interaction of gravity with a Majorana gravitino Ψ_A and a three-index gauge field A_{LMP}. The theory has several continuous symmetries: Local $N = 1$ SUSY, $D = 11$ general covariance, Abelian gauge invariance for the three-form A_{LMP} and of course SO$(10, 1)$ Lorentz invariance. It also has a discrete symmetry associated with the effect of spacetime reflections on the gauge field. This symmetry tells us [18] that the action and equations of motion in eleven dimensions are invariant under an odd number of spatial (or temporal) reflections, together with the reversal of the sign of the gauge field:

$$A_{LMP} \longrightarrow -A_{LMP}$$

In fact, this discrete symmetry is *essential* whenever we consider non-orientable spacetime manifolds in M-theory. This is because we typically think of the four-form F_{LMNP} as being proportional to some volume form, or anti-symmetric tensor ϵ_{LMNP}. It follows that on an non-orientable manifold, F_{LMNP} will not have a definite sign - the sign will change when we propagate around a non-orientable loop. However, propagation around an orientation reversing loop also reflects everything through an odd number of spacetime dimensions, i.e., the equations of motion are still invariant even though the four-form is reduced to the status of a 'pseudo-tensor'. This means that it still makes sense to talk about the eleven dimensional supergravity equations of motion on non-orientable spacetimes.

Now, a key thing to notice is that it really is *not possible* to consistently modify this structure in any way. In particular, the Majorana condition for the gravitino is precisely what one needs in order to match the number of bosonic and fermionic degrees of freedom. One cannot just flippantly introduce other representations for the fermions.

A pleasant feature of life in eleven dimensions is the fact that the real Clifford algebra may be written as

$$Cliff(10, 1; \mathbb{R}) = \mathbb{R}(32)$$

$\mathbb{R}(32)$ denotes the space of real 32×32 matrices and $Cliff(10, 1; \mathbb{R})$ denotes the set of objects γ_ν which satisfy the relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = +2g_{\mu\nu} \tag{20}$$

where $g_{\mu\nu}$ is the metric on eleven dimensional Minkowski space with the signature $(- + + + + + + + + + +)$. In the usual way, these gamma matrices act on a 32 dimensional space of Majorana spinors, which are real with respect to the relevant charge conjugation operator $C_{ij} = -C_{ji}$. Explicitly, such a spinor is just a 32 component column ψ_k, $k = 1, 2, 3, ...32$.

It is *essential* for the contruction of eleven dimensional supergravity that we are able to define, globally and consistently, these Majorana fermions in any eleven dimensional spacetime we wish to consider. Without such spinors, we can have no gravitino field with the right number of degrees of freedom and similarly we cannot define generators of supertranslations in superspace which will transform in the right way.

However, notice that these Majorana/SUSY conditions also select a *unique* fermion type. This is because, once we have made a choice for the representations of P and T, we are not allowed to introduce any complex numbers (this would violate the Majorana condition) and we are not allowed to do any parity doubling (then the fermions would have the wrong number of degrees of freedom for SUSY). But these are the only two mechanisms which we can use to generate other representations for P and T! In other words, there is always only one choice of P and T consistent with the Majorana/SUSY conditions in eleven dimensions.

It would therefore seem that the mathematical structure of M-theory selects a unique pin bundle. Four-dimensional multiplets, the descendants of the unique eleven-dimensional structure, then inherit this choice.

This elegant explanation of how nature may select a unique pin structure is just another example of the power of M-theory

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